Canonical quantization of a minisuperspace model for gravity using self-dual variables*

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The present article summarizes the work of the papers [1] dealing with the quantization of pure gravity and gravity coupled to a Maxwell field and a cosmological constant in presence of spherical symmetry.
Let us stress the following : the motivation for this project was *not* to quantize a black hole. Rather, we regard the present model as an interesting testing ground for the quantization of full 3+1 gravity, in particular by using Ashtekar's self-dual representation.
Throughout we assume that the reader is familiar with the Ashtekar-formulation of gravity ([2]). The conventions are as in [1].

To reduce gravity and our matter content to spherical symmetry, we require that the 3-metric and the Maxwell electric (ϵ^a) and magnetic fields (μ^a) are Lie annihilated by the generators of the SO(3) Killing group. The result of these Killing-reduction prescriptions is the following :
Denoting the angular variables by θ, ϕ, the radial variable by x , and the standard orthonormal basis on the sphere by $\{n^a\}$ we obtain for the gravitational and Maxwell sector respectively ($A_I,\ E^I,\ I=1,2,3$ are angle-independent functions of x and t)

$$\begin{aligned}
(E^x_i, E^\theta_i, E^\phi_i) &= (E^1 n^x_i \sin(\theta), \frac{\sin(\theta)}{\sqrt{2}}(E^2 n^\theta_i + E^3 n^\phi_i), \frac{1}{\sqrt{2}}(E^2 n^\phi_i - E^3 n^\theta_i)), \\
(A^i_x, A^i_\theta, A^i_\phi) &= (A_1 n^x_i, \frac{1}{\sqrt{2}}(A_2 n^\theta_i + (A_3-\sqrt{2})n^\phi_i), \frac{\sin(\theta)}{\sqrt{2}}(A_2 n^\phi_i - (A_3-\sqrt{2})n^\theta_i) \qquad (1)\\
(\epsilon^x, \epsilon^\theta, \epsilon^\phi) &:= (\epsilon(x,t),0,0),\ (\mu^x,\mu^\theta,\mu^\phi) := (\mu(x,t),0,0). \qquad (2)
\end{aligned}$$

The Maxwell potential is thus given by $(\omega_x, \omega_\theta, \omega_\phi) = (\omega(x,t),0,0) + (\Omega_a(x,t,\theta,\phi))$, where Ω_a is a monopole solution with charge μ. The cosmological constant will be labelled by the (real) parameter λ and by performing a 'duality rotation' we get rid of the magnetic charge.
The reality conditions tell us that $E^I,\ i(A_I - \Gamma_I),\ p',\ \omega$ are real where $\Gamma_1 = -(\arctan(E^3/E^2))',\ \Gamma_2 =$

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$-(E^1)'E^3/E,\ \Gamma_3 = (E^1)'E^2/E$ denote the spherically symmetric components of the spin-connection ($E := (E^2)^2 + (E^3)^2$).
The model has 4 canonical pairs ($\omega, p\ ;\ A_I, E^I$) and is subject to the 4 first class constraints (so that the reduced phase space is finite dimensional)

$$\begin{aligned}
{}^M\mathcal{G} &= p' : \text{Maxwell-Gauss constraint,} \\
{}^E\mathcal{G} &= (E^1)' + A_2E^3 - A_3E^2 : \text{Einstein Gauss constraint,} \\
V &= B^2E^3 - B^3E^2 : \text{Vector constraint,} \\
C &= (B^2E^2 + B^3E^3)E^1 + \frac{1}{2}((E^2)^2 + (E^3)^2)(B^1 + \kappa\frac{p^2}{2E^1} + \kappa\lambda E^1) : \text{Scalar constr.}
\end{aligned} \tag{3}$$

where we have abbreviated the components of the gravitational magnetic fields by B^I ($B^1 = 1/2((A_2)^2 + (A_3)^2),\ B^2 = (A_3)' + A_1A_2,\ B^3 = -(A_2)' + A_1A_3$)).

For spherically symmetric systems, the topology of the 3 manifold is necessarily of the form $\Sigma^{(3)} = S^2 \times \Sigma$ where Σ is the 1-dimensional manifold (restrictinig ourselves to asymptotically flat topologies in which case suitable boundary conditions on the fields are imposed)) $\Sigma = \Sigma_n\ ,\ \Sigma_n \cong K \cup \bigcup_{A=1}^n \Sigma_A$, i.e. the hypersurface is the union of a compact set K and a collection of asymptotic regions (each of which is diffeomorphic to the positive real line without a compact interval) with outward orientation and all of them are joined to K. The case of two ends is physically most interesting.

We now apply the method of symplectic reduction ([3]). Choosing 'cylinder coordinates' $(A_2, A_3) = \sqrt{A}(\cos(\alpha), \sin(\alpha)),\ (E^2, E^3) = \sqrt{E}(\cos(\beta), \sin(\beta))$ it is easy to see that the Gauss-reduced symplectic potential becomes

$$i\kappa\Theta[\partial_t] = \int_\Sigma dx(\dot{\gamma}\pi_\gamma + \dot{B}^1\pi_1 + \dot{\omega}p(i\kappa))\ , \tag{4}$$

where $\gamma := A_1 + \alpha', \pi_\gamma := E^1, B^1 := \frac{1}{2}(A-2)$ and $\pi_1 := \sqrt{E/A}\cos(\alpha - \beta)$.
In the following p will already be taken as a constant. Also we will deal with an arbitrary cosmological constant for the sake of generality.
We take then the following linear combinations of the vector and the scalar constraint functional $E^1E^2V + E^3C$ and $-E^1E^3V + E^2C$ and set these expressions strongly zero. For non-degenerate metrics ($E \neq 0$) we can now solve for $E^2,\ E^3$ and insert this solution into the Gauss constraint. The final solution is given by

$$[\kappa(-p^2 + \lambda(E^1)^2/3) + B^1E^1]^2 = m^2E^1\ . \tag{5}$$

The integration constant, m, is real and can be shown to coincide with the gravitational mass up to a factor.

Equation (5) is an algebraic equation of fourth order in terms of E^1 and therefore very unpractical to handle.
The idea is to change the polarization and to chose B^1 as a momentum. Then (5) can be easily solved for B^1 and the vector constraint for γ. The result for the reduced symplectic potential is given by (modulo a total differential)

$$\Theta[\partial_t] = \dot{p}\int_\Sigma dx(-i\frac{p\pi_1}{\pi_\gamma} - \omega) + \dot{m}\int_\Sigma dx(-i/\kappa)\frac{\pi_1}{\sqrt{\pi_\gamma}} =: \sum_A [\dot{\Phi}_A p_A + \dot{T}_A m_A]. \qquad (6)$$

It can be shown that π_1 is imaginary while $\pi_\gamma = E^1$ is real. Accordingly, T and Φ are both real and resulting reduced phase space can be described as follows : in every asymptotic end A we have a cotangent bundle over R^2. The treatment for K is similar.

One can explicitly check that the variables m, T, p, Φ commute with all the constraints for Lagrange multipliers of comact support, that is, they are Dirac observables.
The reduced action takes the form $S_{red} = \sum_A [\dot{T}_A m_A + \dot{\Phi}_A p_A - H_{red;A}]$ where $H_{red;A} = N_a m_A + U_A p_A$ is the reduced Hamiltonian and we have defined $N_A(t) := N(x = \partial\Sigma_A, t)$, $U_A(t) := U(x = \partial\Sigma_A, t)$ where $N := \det(q)^{1/2}\underset{\sim}{N}$ is the lapse function, $G := \int dx[\Lambda({}^E\mathcal{G}) + U({}^M\mathcal{G}) + N^x V + \underset{\sim}{N}C]$ being the constraint generator. The solution of the equations of motion can be written ($\frac{d}{dt}\tau_A = N_A$ and $\frac{d}{dt}\phi_A = U_A$)

$$m_A(t) = \text{const.},\; T_A(t) = \text{const.} + \tau_A(t),\; p_A(t) = \text{const.},\; \Phi_A(t) = \text{const.} + \phi_A(t) \qquad (7)$$

i.e. the reduced system adopts the form of an integrable system where the role of the action variables is played by the masses and the charges whereas their conjugate variables take the role of the angle variables.
What now is the interpretation of this second set of conjugate variables ? The interpretation of m and p follows simply from the fact that they can be derived from the reduced Hamiltonian, i.e. they are the well-known surface integrals ADM-energy and Maxwell-charge. However, their conjugate partners are genuine volume integrals and we are not able to write them as known surface integrals. The solution (7) allows for the interpretation that T is the eigentime at spatial infinity while Φ plays the same role as the variable conjugate to the electric charge of 1+1 Maxwell theory.

We finally come to the quantization of the system. We follow the group theoretical quantization scheme ([4]).
The phase space for every end is just the cotangent bundle over the two-plane so the unique Hilbert-space is the usual one : $L_2(R^2, d^2x)$. The Schroedinger- equation in the polarization in which eigentime and the flux act by multiplication and the mass and the charge by differentiation

becomes unambiguously

$$i\hbar\frac{\partial}{\partial t}\Psi(t;\{T_A\},\{\Phi_A\}) = (-i\hbar\sum_{A=1}^{n}[N_A(t)\frac{\partial}{\partial T_A}+U_A(t)\frac{\partial}{\partial\Phi_A}])\Psi(t;\{T_A\},\{\Phi_A\})\ . \tag{8}$$

It can be solved trivially by separation : $\Psi(t;\{m_A\},\{\Phi_A\}) := \prod_{A=1}^{n}\psi_A(t,m_A,\Phi_A)$ and by using the functions $\tau_A,\ \phi_A$:

$$\psi_A(t,T_A,\phi_A) = C_A\exp(k_A\frac{i}{\hbar}[T_A-\tau_A(t)])\times\exp(l_A\frac{i}{\hbar}[\Phi_A-\phi_A(t)]) \tag{9}$$

where C_A is a complex number, whereas k_A, l_A must be real because the spectrum of the momenta, which are self-adjoint, is real.

Final remark :

It is interesting to express the observables found in terms of the Ashtekar-connection : Restricting to the case $\lambda = 0$ we obtain for the integrand of the variables T and $\Phi+\int_\Sigma dx\omega$ respectively

$$-2\frac{A_1+[\arctan(\frac{A_3}{A_2})]'}{(2B^1)^{2-n}}\frac{[p^2\kappa+\frac{m^2}{B^1}+\sqrt{[p^2\kappa+\frac{m^2}{B^1}]^2-[p^2\kappa]^2})]^{2-n}}{p^2\kappa+1/2(\frac{m^2}{B^1}+\sqrt{[p^2\kappa+\frac{m^2}{B^1}]^2-[p^2\kappa]^2})} \tag{10}$$

where $n = 1/2$ and $n = 1$ respectively and $B^1 = 1/2((A_2)^2+(A_3)^2-2)$.

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